# Synchronization of coupled oscillators

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29 nov. 2011

# The Kuramoto model

The equation

$$\dot{\theta}_j = \omega_j + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_j), \quad j = 1, \ldots, N$$

is the Kuramoto model with parameters

$\omega_j$: frequency of each oscillator
$K$: coupling strength
$N$: number of oscillators

$\theta_j$ is a phase variable in $[0, 2\pi]$.

# Initial conditions

- Choose $N$ random frequencies from a distribution $g(\omega)$

Typically, $g(\omega)$ is unimodal (Gaussian or Lorentzian)

- Choose inital phases (e.g. randomly, or constant, $\theta_j = 0, \ \forall j$)

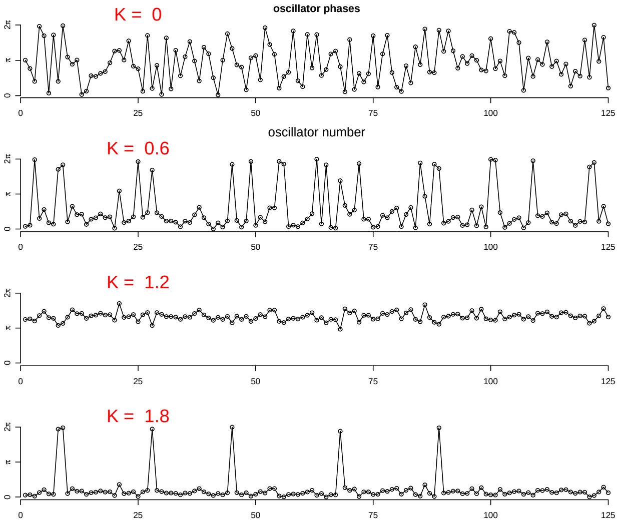
# Coupling topologies

All-to-all coupling: $\mathcal{O}(N^2)$
Local coupling on a circle
"Small world networks" — random couplings with given probability

**Further variations:**
Delayed couplings (neural nets)
Different populations of oscillators (different natural frequencies)
External driving: periodic, quasi-periodic, noise, etc.

# Increasing $K$

As the coupling increases, the oscillators synchronize.



# The order parameter

Phases are taken modulo $2\pi$. Use *circular statistics*.

$$re^{i\psi} = \frac{1}{N}\sum_{k=1}^{N} e^{i\theta_k}$$

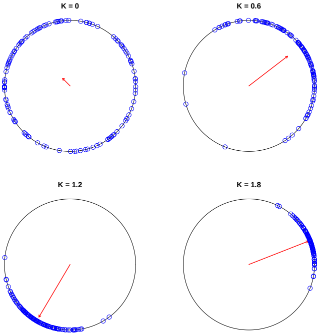
Order parameter: a complex number in polar form

$\psi \in [0, 2\pi]$ is the "average phase"
$r \in [0, 1]$ is the phase coherence

Oscillators in sync $\rightarrow r \approx 1$
independent / unordered $\rightarrow r \approx 0$

# Order parameter geometry

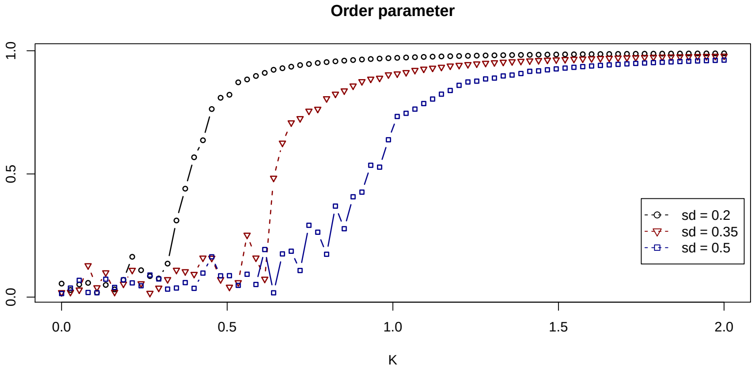
The order parameter is a vector.
Its length increases as oscillators synchronize.




Red arrow: order parameter
Blue circles: phase values

# Onset of sync — critical coupling

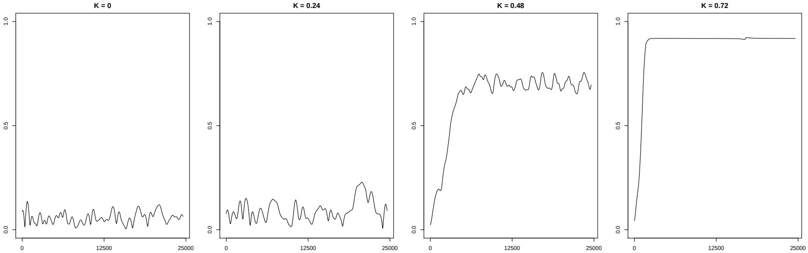
For $K < K_c$, $r \approx 0$
For $K > K_c$, $r$ approaches 1.




$K_c$ depends on the spread of the frequency distribution $g(\omega)$

# Time evolution of the order parameter

Coupling dependece:
For $K > K_c$ the order parameter rapidly increases.




$N = 225$ oscillators, 25000 time steps.

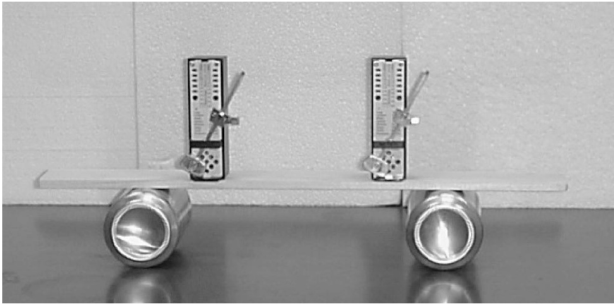
# Advanced techniques

The Kuramoto model is studied in the limit $N \to \infty$.
Continuous probability distribution $\rho(\theta)$ of phase
A partial differential equation describes the evolution of $\rho(\theta)$
Variable substitutions: The order parameter is explicitly used
Rescaling of variables: let $\langle g(\omega) \rangle = 0$
The **Ott-Antonsen ansatz** (2008): the dynamics of the Kuramoto model can be reduced to a low-dimensional ODE.

# Phenomena in coupled oscillators

- Phase locking
- Incoherent to synchronized transition
- Oscillation death
- Chimera states
- Noise-induced synchronisation

Examples:
phase locking and oscillation death in organ pipes
period doubling in applause (Néda et al. 2000)



# Landau-Stewart oscillators

The Hopf oscillator (or Landau-Stewart oscillator):

$$\dot{z} = (\alpha + i\omega - |z|^2)z$$

for complex z.
Limit cycle oscillator with amplitude $\sqrt{\alpha}$ if $\alpha > 0$, and frequency $\omega$.

# Coupled Landau-Stewart oscillators

Simplest case of mean field coupling:

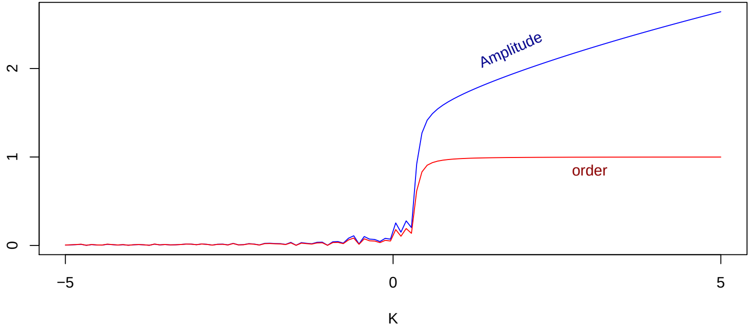
$$\dot{z}_j = (\alpha + i\omega_j - |z_j|^2)z_j + \frac{K}{N}\sum_{k=1}^{N} z_k, \quad j = 1, \ldots, N$$

$K < 0$ is inhibitory
$K > 0$ increases amplitude
Reduces to the Kuramoto model if the amplitude is neglected (Pyragas et al. 2007).

# Order parameter

Define the order parameter for the Landau-Stewart oscillator

$$re^{i\psi} = \frac{1}{N}\sum_{k=1}^{N} e^{i \arg z_k}$$




Average amplitude and order ($r$) for 125 oscillators

# External forcing

Adaptive oscillators for beat tracking uses external forcing, e.g.

$$\dot{z}_j = f(z_j) + K\bar{Z} + x(t)$$

$f(z)$ is the Landau-Stewart oscillator
$\bar{Z}$ is the average of oscillators
$x(t)$ is the input signal

# Sync in chaotic systems

Discovered by Pecora & Carroll (1990)

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + K y_1 \\ \dot{\mathbf{y}} &= \mathbf{f}(\mathbf{y}) + K x_1 \end{aligned}$$

where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ or higher dimension. Criterion of sync: $\|\mathbf{x} - \mathbf{y}\| < \epsilon$
Generalized synchronization (of phase): $\dot{\theta}_1 = \dot{\theta}_2$ — same frequency, phase may differ.

# Further reading

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